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USE OF AN ORTHOEXPONENTIAL TRANSFORMATION IN THE STUDY OF THE NON-STATIONARY EQUATIONS OF THERMOELASTICITY AND THERMOVISCOELASTICITY*

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A system of orthoexponential polynomials (OEP) orthogonal in the interval $t \in [0, \infty)$ representing a special case of the orthoexponential Jacobi polynomials /1/ is studied. It is proposed to use the OEP as the kernels of an integral transformation (the OEP transformation) in time, since, compared with Laplace transformations, its use simplifies the procedure for obtaining the originals of the quantities required. The OEP transformation is used to solve the non-stationary equations of thermoelasticity and thermoviscoelasticity. The initial equations are reduced to the corresponding systems of ordinary triangular differential equations, and their general solutions are constructed.

1. *Definition*. The polynomials $\mathrm{oep}_n(\varkappa, t)$ ($\varkappa$ is a non-negative number and $n = 0, 1, 2, \ldots$), specified in a semi-infinite interval $t \in [0, \infty)$ by the relations

$$\mathrm{oep}_n(\varkappa, t) = \sum_{k=0}^{n} b_{nk} e^{-kt}; \quad b_{nk} = (-1)^{n+k} \binom{n}{k} \binom{n+k+\varkappa+1}{n} \tag{1.1}$$

are called orthoexponential polynomials (OEP).

The polynomials $\mathrm{oep}_n(\varkappa, t)$ are a special case of the Jacobi OEP $p_n(\alpha, \beta, t)$ for $\alpha = 0$ and $\beta = \varkappa - 1$, whose expressions follow from the classical Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ /2/ after the following change of the argument $x = 2e^{-t} - 1$, $\alpha > -1$, $\beta > -1$, i.e.

$$\mathrm{oep}_n(\varkappa, t) == p_n(0, \varkappa - 1, t) = P_n^{(0, \varkappa-1)}(2e^{-t} - 1) \tag{1.2}$$

We will introduce into the discussion the class of functions $L_{2,w}[0, \infty)$ belonging to Hilbert space, for which the scalar product is given by the relation

$$(f \cdot g) = \int w(t) f(t) g(t)\, dt$$

where $w(t)$ is a function positive in $t \in [0, \infty)$. Here and henceforth the integration is carried out from zero to infinity, unless stated otherwise. The real-valued function $f(t)$, $t \in [0, \infty)$ belongs to $L_{2,w}[0, \infty)$, provided that the condition $\int f^2(t) w(t) dt < \infty$ holds.

Since the Jacobi OEP are complete and closed on the interval $t \in [0, \infty)$ and relations (1.2) hold, it follows that $\mathrm{oep}_n(\varkappa, t)$ $(n = 0, 1, 2, \ldots)$ form an orthogonal system of functions with weight $w(t) = e^{-\varkappa t}$, $\varkappa > 0$, complete and closed in the class of functions $L_{2,w}[0, \infty)$.

Let us state some properties of the polynomials $\mathrm{oep}_n(\varkappa, t)$ whose validity follows from the analogous properties of the polynomials $p_n(\alpha, \beta, t)$ and $P_n^{(\alpha,\beta)}(x)$:

$$\int e^{-\varkappa t} \mathrm{oep}_n(\varkappa, t)\, \mathrm{oep}_k(\varkappa, t)\, dt = \frac{\delta_{nk}}{\varkappa + 2n}; \quad \delta_{nk} = \begin{cases} 0, & n \neq k \\ 1, & n = k \end{cases} \tag{1.3}$$

$$\mathrm{oep}_0(\varkappa, t) \equiv 1, \quad \mathrm{oep}_1(\varkappa, t) = (\varkappa + 1) e^{-t} - \varkappa \tag{1.4}$$

$$\mathrm{oep}_{n+1}(\varkappa, t) = (A_n e^{-t} - B_n)\, \mathrm{oep}_n(\varkappa, t) - D_n\, \mathrm{oep}_{n-1}(\varkappa, t), \quad n = 1, 2, \ldots$$

$$A_n = \frac{(\varkappa + 2n)(\varkappa + 2n + 1)}{(n+1)(\varkappa + n)}, \quad B_n = \frac{(\varkappa + 2n)[(\varkappa + n)^2 + n^2 - \varkappa]}{(n+1)(\varkappa + n)(\varkappa + 2n - 1)}$$

$$D_n = \frac{n(\varkappa + n - 1)(\varkappa + 2n + 1)}{(n+1)(\varkappa + n)(\varkappa + 2n - 1)}$$

$$\mathrm{oep}_n(\varkappa, 0) = 1; \quad \lim_{t \to \infty} \mathrm{oep}_n(\varkappa, t) = (-1)^n \binom{n + \varkappa - 1}{n} = b_{n0}, \quad n = 0, 1, 2, \ldots \tag{1.5}$$

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$$\mathrm{oep}_n(\varkappa, t) = \frac{(-1)^n}{n!} e^{-(\varkappa-1)t} \left\{ \frac{d^n}{d\xi^n} [(1-\xi)^n \xi^{\varkappa+n-1}] \right\} \Big|_{\xi=e^t}, \quad n = 1, 2, \ldots \tag{1.6}$$

((1.3) is the orthogonality relation, (1.4) are the recurrence relations, (1.5) are the boundary values and (1.6) is an analogue of the Rodrigues formula).

The following differentiation formula holds for $\mathrm{oep}_n(\varkappa, t)$ $(n = 0, 1, 2, \ldots)$ (this is proved by mathematical induction using the relations (1.4)):

$$\frac{d}{dt} \mathrm{oep}_0(\varkappa, t) = 0 \tag{1.7}$$

$$\frac{d}{dt} \mathrm{oep}_n(\varkappa, t) = -n\, \mathrm{oep}_n(\varkappa, t) - \sum_{k=0}^{n-1} (\varkappa + 2k)\, \mathrm{oep}_k(\varkappa, t), \quad n = 1, 2, \ldots$$

The expressions for the higher-order derivatives of OEP follow from (1.7).

$$\frac{d^k}{dt^k} \mathrm{oep}_n(\varkappa, t) = (-n)^k \mathrm{oep}_n(\varkappa, t) - \Phi_{kn}, \quad \begin{matrix} k = 0, 1, 2, \ldots \\ n = 1, 2, \ldots \end{matrix} \tag{1.8}$$

$$\Phi_{0n} \equiv 0, \quad \Phi_{1n} = \sum_{m=0}^{n-1} (\varkappa + 2m)\, \mathrm{oep}_m(\varkappa, t)$$

$$\Phi_{kn} = -n\Phi_{k-1,n} + \sum_{m=0}^{n-1} (\varkappa + 2m) [(-m)^{k-1} \mathrm{oep}_m(\varkappa, t) + (-1)^{k-1} \Phi_{k-1,m}]$$

Assertion. The polynomial $\mathrm{oep}_n(\varkappa, t)$ has n simple zeros in the semi-infinite interval $t \in [0, \infty)$.

Proof. Let us assume that the polynomial $\mathrm{oep}_n(\varkappa, t)$ changes its sign within the interval on passing through k points. It is clear that $0 \leqslant k \leqslant n$. The Assertion will be proved, if $k = n$. Let us consider the function

$$Q_k(t) = \begin{cases} 1, & k = 0 \\ \prod_{j=0}^{k} (e^{-t} - e^{-t_j}), & 0 < k \leqslant n, \quad t \in [0, \infty) \end{cases}$$

Here t_k are points, on passing through which the polynomial $\mathrm{oep}_n(\varkappa, t)$ changes its sign. Clearly, the product $Q_k(t)\, \mathrm{oep}_n(\varkappa, t)\, e^{-\varkappa t}$ does not change its sign in the interval $t \in [0, \infty)$, therefore

$$\int e^{-\varkappa t} Q_k(t)\, \mathrm{oep}_n(\varkappa, t)\, dt \neq 0$$

From this it follows that $k = n$, since when $k < n$, we have

$$\int e^{-\varkappa t} \mathrm{oep}_n(\varkappa, t)\, e^{-kt} dt = 0$$

Let us consider the problem of expanding a function belonging to $L_{2,w}[0, \infty)$, where $w(t) = e^{-\varkappa t}$, $\varkappa > 0$, in a series in OEP. In /1/ Rau's theorem /3/ was used to prove the theorem for expanding the function $f(t) \in L_{2,w}[0, \infty)$, $w(t) = e^{-(\beta+1)t}(1 - e^{-t})^\alpha$, $\alpha > -1$, $\beta > -1$ in a series in $p_n(\alpha, \beta, t)$. By analogy with this theorem, we shall now formulate a theorem for expanding a function belonging to $L_{2,w}[0, \infty)$, $w(t) = e^{-\varkappa t}$ in a series in terms of the polynomials $\mathrm{oep}_n(\varkappa, t)$.

Theorem 1. Let a function $f(t)$, bounded and continuous in $t \in [0, \infty)$, have a piecewise continuous derivative which satisfies the condition $\lim e^t f'(t) < \infty$ as $t \to \infty$. Then the series

$$\sum_{n=0} f_n (\varkappa + 2n)\, \mathrm{oep}_n(\varkappa, t) \quad \left(f_n = \int f(t) e^{-\varkappa t} \mathrm{oep}_n(\varkappa, t)\, dt\right)$$

will converge, for $\varkappa > 0$, uniformly to $f(t)$ in every closed interval $t \in [t_1, t_2]$ $0 < t_1 < t_2 < \infty$.

A simple, and therefore important special practical case of the class of OEP, is the case $\varkappa = 0$.

2. An orthoexponential transformation (an OEP transformation) of a function belonging to $L_{2,w}[0, \infty)$, $w(t) = e^{-\varkappa t}$ is described by the following pair of relations:

$$\text{OЭП}\{f\} \equiv f_n = \int e^{-\varkappa t} f(t)\, \mathrm{oep}_n(\varkappa, t)\, dt, \quad f(t) = \sum_{n=0} (\varkappa + 2n), f_n\, \mathrm{oep}_n(\varkappa, t) \tag{2.1}$$

We shall call the function $f(t)$ the original function, or simply the original, and f_n is the mapping of the function $f(t)$.

The corresponding transformation for $\varkappa = 0$ was used successfully when studying non-stationary processes in elastic and viscoelastic media under the action of impulsive forces*.
(*Mokrik R.I. and Oliyarnik I.V. Combined non-stationary problem of thermoviscoelasticity for

a half-space. Deposited in UkrNIINTI, 1569, Lvov, 1984).

The idea of obtaining the original in the form of a series in a system of orthogonal functions was used in the Laguerre transformation /4/, and in the finite-dimensional integral transformations. There exists, however, an essential difference in the method of utilizing this idea in the latter transformations and in the Laguerre transformations (see (2.1)).

Let us give some properties of the transformation (2.1). In what follows, we shall assume that $f(t)$ and $g(t)$ are continuously differentiable functions which satisfy the conditions of Theorem 1.

Linearity. If f_n is the mapping of the function $f(t)$ and g_n of the function $g(t)$, then $F_n = af_n + bg_n$ will be the mapping of the function $F(t) = af(t) + bg(t)$ where a, b are constants.

Mapping of derivative of a function. Let f_n be the mapping of the function $f(t)$. Then there exists a mapping of the function $F(t) = f'(t)$ given by the expression

$$F_n = (\varkappa + n) f_n + \sum_{k=0}^{n-1} (\varkappa + 2k) f_k - f(0) \tag{2.2}$$

Integration of the original. Let f_n be the mapping of the function $f(t)$. Then the mapping F_n of the function

$$F(t) = \int_0^t f(x)\,dx$$

will be given by the expression

$$F_n = (\varkappa + n)^{-1} \left[f_n - \sum_{k=0}^{n-1} (\varkappa + 2k) F_k\right] \tag{2.3}$$

Forumulas (2.2) and (2.3) are proved using relations (2.1) and integration by parts taking relations (1.7) into account.

Mapping a convolution of two functions. Let f_n be the mapping of the function $f(t)$, and let g_n be the mapping of the function $g(t)$. Then there exists a mapping h_n of the function

$$h(t) = \int_0^t f(t-\xi)\, g(\xi)\, d\xi$$

given by the expression

$$h_n = b_{nn}^{-1} \sum_{i=0}^{n} f_i \alpha_{ni} \sum_{i=0}^{n} g_i \alpha_{ni} - \sum_{k=0}^{n-1} \alpha_{nk} h_k \tag{2.4}$$

$$\alpha_{nk} = b_{nk} \sum_{i=0}^{k} (\varkappa + 2i)\, \alpha_{ki}^*$$

$$\alpha_{k0}^* = (\varkappa + k)^{-1}, \quad \alpha_{ki}^* = \prod_{j=0}^{i-1} (k-j) \Big/ \prod_{j=0}^{i} (\varkappa + k + j), \quad i \geqslant 1$$

where b_{nn} is given by (1.1).

Proof. The function $h(t)$ is continuously differentiable and satisfies the conditions of Theorem 1. This follows from the corresponding properties of the functions $f(t)$ and $g(t)$. Therefore h_n exists.

According to the definition (2.1) we have

$$h_n = \int e^{-\varkappa t} \operatorname{oep}_n(\varkappa, t)\, dt \int_0^t f(t-\xi)\, g(\xi)\, d\xi = \int e^{-\varkappa t} \operatorname{oep}_n(\varkappa, t)\, dt \int f(t-\xi)\, g(\xi)\, H(t-\xi)\, d\xi$$

where $H(t)$ is the Heaviside unit function. Changing the order of integration and following this by changing the variables in the inner integral, we obtain

$$h_n = \int g(\xi)\, e^{-\varkappa\xi} \sum_{k=0} (\varkappa + 2k)\, f_k L_{nk}(\varkappa, \xi)$$

$$L_{nk}(\varkappa, \xi) = \int e^{-\varkappa t}\, \mathrm{oep}_n(\varkappa, t+\xi)\, \mathrm{oep}_k(\varkappa, t)\, dt = \sum_{j=0}^{n-k} a_j^{(n,k)}(\varkappa)\, e^{-(n-j)\varkappa}$$

We note that $L_{nk}(\varkappa, \xi) = 0$ for $k > n$. Substituting the value of $L_{nk}(\varkappa, \xi)$ into the expression for h_n and using the expression

$$e^{-m\xi} = \sum_{i=0}^{m} \alpha_{mi}^{*}\, \mathrm{oep}_i(\varkappa, \xi)$$

we obtain the relation (2.4).

3. We shall illustrate the possibilities of an integral transformation on the equations of thermoelasticity and thermoviscoelasticity. Small perturbations in a certain unperturbed thermoelastic or thermoviscoelastic medium are described by the following system of equations:

$$L_1 \operatorname{grad} \operatorname{div} \mathbf{u} - L_2 \operatorname{rot} \operatorname{rot} \mathbf{u} - L_3 \operatorname{grad} T = L_4 (\rho \partial_t^2 \mathbf{u} - \mathbf{F}) \tag{3.1}$$
$$\Delta T - a^{-1} l \partial_t T - L_5 l \operatorname{div}(\partial_t \mathbf{u}) + \lambda_T^{-1} l W = 0,\ l = 1 + t_r \partial_t$$

Here $\mathbf{u}$ is the displacement vector, T is the change in the absolute temperature of the medium, ρ is its density, $\mathbf{F}$ is the mass force vector, W is a function of the heat sources, λ_T is the thermal conductivity, a is the thermal diffusivity, and t_r is the relaxation time of the heat flux /5/.

For a thermoelastic medium we have

$$L_1 = \lambda_0 + 2\mu_0,\quad L_2 = \mu_0,\quad L_3 = \gamma_0 = \alpha_T (3\lambda_0 + 2\mu_0),\quad L_4 \equiv 1, \tag{3.2}$$
$$L_5 = \eta_0 = \gamma_0 T_0 \lambda_T^{-1}$$

(λ_0, μ_0 are the Lamé coefficients, α_T is the coefficient of linear thermal expansion of the medium, and T_0 is the initial temperature of the medium).

For a thermoviscoelastic medium $L_1, \ldots, L_5$ are operators of the form

$$L_1\varphi = (\lambda + 2\mu)\cdot\varphi,\quad L_2\varphi = \mu\cdot\varphi,\quad L_3\varphi = \alpha_T\ (3\lambda + 2\mu)\cdot\varphi,\quad L_4\varphi = \varphi \tag{3.3}$$
$$L_5\varphi = \alpha_T \lambda_T^{-1} T_0 (3\lambda + 2\mu)\cdot\varphi$$
$$\lambda\cdot\varphi = \lambda_0\Big[\varphi - \int_0^t \lambda(t-\xi)\varphi(\xi)\,d\xi\Big],\quad \mu\cdot\varphi = \mu_0\Big[\varphi - \int_0^t \mu(t-\xi)\varphi(\xi)\,d\xi\Big]$$

or

$$L_1 = \frac{2}{3}(\bar P_1 \bar P_4 + 2\bar P_2 \bar P_3),\quad L_2 = \bar P_2 \bar P_3,\quad L_3 = 2\bar P_1 \bar P_4,\quad L_4 = 2\bar P_1 \bar P_3, \tag{3.4}$$
$$L_5 = T_0 \lambda_T^{-1} L_3,\quad \bar P_i = \sum_{k=0}^{N_i} a_k^{(i)} \partial_t^{\,k},\quad i = 1, 2, 3, 4;\quad \partial_t^{\,k} = \frac{\partial^k}{\partial t^k}$$

In order to simplify the algebra we shall restrict ourselves to considering the case of axisymmetric perturbations in the media in question, in a cylindrical system of coordinates (r, θ, z). Then the displacement vector $\mathbf{u}(u_r, 0, u_z)$ and temperature T will be functions of time t and of two spatial coordinates r and z. Let us write the displacement vector $\mathbf{u}$ in the form

$$\mathbf{u} = \operatorname{grad} \Phi + \operatorname{rot} \boldsymbol{\Psi};\quad \Phi \equiv \Phi(t, r, z),\quad \boldsymbol{\Psi} \equiv (0, \Psi, 0)\quad (t, r, z)$$

(Φ and $\boldsymbol{\Psi}$ are the scalar and vector potentials). In regions free of mass forces and sources ($F \equiv 0$, $W \equiv 0$), Eqs.(3.1) can be reduced to

$$L_1 \Delta\Phi - L_4 \rho \partial_t^2 \Phi = L_3 T,\quad L_2(\Delta\Psi - r^{-2}\Psi) - L_4 \rho \partial_t^2 \Psi = 0$$
$$\Delta T - a^{-1} l \partial_t T + L_5 l \partial_t \Delta\Phi = 0\quad (\Delta = \partial_r^2 + r^{-1}\partial_r + \partial_z^2) \tag{3.5}$$

Let us carry out, in (3.5), the OEP transformation (2.1) with respect to time t, and the Handel transformation with respect to the radial coordinate r. Taking into account the properties of the above transformations, we obtain (a prime denotes a derivative with respect to z)

$$\Phi_n'' - p_n \Phi_n - \gamma_n T_n = \sum_{k=0}^{n-1} F_1(T_k, \Phi_k, \Phi_k''),\quad \Psi_n'' - s_n \Psi_n = \sum_{k=0}^{n-1} F_2(\Psi_k, \Psi_k'') \tag{3.6}$$

$$T_n'' - q_nT_n + \eta_n(\Phi_n'' - \xi^2\Phi_n) = \sum_{k=0}^{n-1} F_3(T_k, \Phi_k, \Phi_k'')$$

$$q_n = \xi^2 + a^{-1}t_{rn}, \quad t_{rn} = (\varkappa + n)\,[1 + (\varkappa + n)\,t_r]$$

$$\zeta_{nk} = (\varkappa + 2k)\,[1 + (\varkappa + n + k)\,(n - k)\,t_r], \quad \omega_{nk} = (\varkappa + 2k)\,(n - k)\cdot(\varkappa + n + k)$$

The functions $\Phi_n(\xi, z)$, $\Psi_n(\xi, z)$, $T_n(\xi, z)$ $(n = 0, 1, 2, \ldots)$ are given by the formula

$$(\Phi_n, \Psi_n, T_n)(\xi, z) = \iint e^{-\varkappa t}\,\mathrm{oep}_n(\varkappa, t)\, rJ_i(\xi r)\,(\Phi, \Psi, T)(t, r, z)\,dtdr \tag{3.7}$$

$i = 0$ for Φ, T, $i = 1$ for Ψ; ξ is the parameter of the Hankel transformation and $J_i(\xi r)$ is an i-th order Bessel function of the first kind.

The values of the remaining quantities are:

$$p_n = \xi^2 + (\varkappa + n)^2/c_1^2, \quad \gamma_n = \gamma_0/(\lambda_0 + 2\mu_0), \quad s_n = \xi^2 + (\varkappa + n)^2/c_2^2$$

$$\eta_n = \eta_0 t_{rn}, \quad F_1(\ldots) = \omega_{nk}\Phi_k(\xi, z)/c_1^2, \quad F_2(\ldots) = \omega_{nk}\Psi_k(\xi, z)/c_2^2$$

$$F_3(\ldots) = \zeta_{nk}\,[a^{-1}T_k(\xi, z) - \eta_0(\Phi_k''(\xi, z) - \xi^2\Phi_k(\xi, z))]$$

in the case of thermoelasticity,

$$p_n = \xi^2 + \frac{(\varkappa + n)^2}{c_1^2(1 - g_{nn})}, \quad \gamma_n = \beta_0 - \beta^*, \quad \beta_0 = \frac{\gamma_0}{\lambda_0 + 2\mu_0}$$

$$\beta^* = \frac{\alpha_{nn}\alpha_T(3\lambda^* + 2\mu^*)}{(\lambda_0 + 2\mu_0)\,b_{nn}}, \quad s_n = \xi^2 + \frac{(\varkappa + n)^2}{c_2^2(1 - \chi_{nn})}, \quad \eta_n = (\eta_0 - \eta^*)\,t_{rn}$$

$$\eta^* = \alpha_T T_0 \frac{3\lambda^* + 2\mu^*}{\lambda_T}, \quad \lambda^* = \lambda_0 \sum_{k=0}^{n} \alpha_{nk}\lambda_k^*, \quad \mu^* = \mu_0 \sum_{k=0}^{n} \alpha_{nk}\mu_k^*$$

$$(\mu_k^*, \lambda_k^*) = \int e^{-\varkappa t}\,\mathrm{oep}_k(\varkappa, t)\,(\mu, \lambda)(t)\,dt$$

$$F_1(\ldots) = [(\omega_{nk}/c_1^2 - \xi^2 g_{nk})\,\Phi_k + g_{nk}\Phi_k'' - \beta^* T_k]\,(1 - g_{nn})^{-1}$$

$$F_2(\ldots) = [(\omega_{nk}/c_2^2 - \xi^2\chi_{nk})\,\Psi_k + \chi_{nk}\Psi_k'']\,(1 - \chi_{nn})^{-1}$$

$$F_3(\ldots) = \zeta_{nk}\,[a^{-1}T_k - \eta_0(\Phi_k'' - \xi^2\Phi_k) + \alpha_T T_0(3\lambda_0 H_k + 2\mu_0 G_k)\,\lambda_T^{-1}] + \eta^* t_{rn}\alpha_{nk}(\Phi_k'' - \xi^2\Phi_k)\,b_{nn}^{-1}$$

$$g_{nk} = \frac{\alpha_{nk}(\lambda^* + 2\mu^*)}{b_{nn}(\lambda_0 + 2\mu_0)}, \quad \chi_{nk} = \frac{\alpha_{nk}\mu^*}{b_{nn}\mu_0}$$

$$H_k = b_{kk}^{-1} \sum_{i=0}^{k} \alpha_{ki}(\Phi_i'' - \xi^2\Phi_i) \sum_{i=0}^{k} \alpha_{ki}\lambda_i^*$$

$$G_k = b_{kk}^{-1} \sum_{i=0}^{k} \alpha_{ki}(\Phi_i'' - \xi^2\Phi_i) \sum_{i=0}^{k} \alpha_{ki}\mu_i^*$$

in the case of thermoviscoelasticity for the relations (3.3), and

$$p_n = \xi^2 + \rho \sum_{i=0}^{N_1+N_4} A_i(\varkappa + n)^{i+2}\Big[\sum_{i=0}^{M} B_i(\varkappa + n)^i\Big]^{-1}$$

$$\gamma_n = \sum_{i=0}^{N_1+N_4} C_i(\varkappa + n)^i\Big[\sum_{i=0}^{M} B_i(\varkappa + n)^i\Big]^{-1}, \quad \eta_n = [1 + (\varkappa + n)\,t_r] \sum_{i=0}^{N_1+N_4} C_i(\varkappa + n)^i$$

$$s_n = \xi^2 + 2\rho \sum_{i=0}^{N_1} a_i^{(1)}(\varkappa + n)^{i+2}\Big[\sum_{i=0}^{N_2} a_i^{(2)}(\varkappa + n)^i\Big]^{-1}$$

$$M = \max\,\{N_1 + N_4, N_2 + N_3\}$$

for relations (3.4).

The form of the functions $F_j(\ldots)$ $(j = 1, 2, 3)$ for a specific model of a viscoelastic body is determined in accordance with (1.8), (2.2) and (3.4), and the constants A_i, B_i and C_i are expressed in terms of the constants of the model of a viscoelastic body.

Thus we have reduced the problem of solving the initial system of Eqs.(3.5) to that of solving a sequence of inhomogeneous ordinary differential equations. We note that the right-hand sides of (3.5) (the inhomogeneous terms) represent a combination of solutions of the previous equations. Finally we add, that in the case of non-zero initial conditions the right-hand sides of Eqs.(3.6) will contain terms determining the values of the quantities required, and of their derivatives at the initial instant.

When the domain of variation of the radial variable r is finite, then a finite-dimensional Hankel transformation should be used instead of the Hankel transformation (3.7). The end result is again a sequence of inhomogeneous ordinary differential equations.

4. Let us now pass to the problem of constructing a general solution of the system of Eqs.(3.6), which represents a special case of the infinite triangular system of ordinary differential equations. A theorem of the existence of a system of fundamental solutions of such systems was proved in /6/.

Unlike in the triangular type systems discussed in /4/, the differential operators on the left-hand sides of (3.6) depends on the parameter n. This particular property makes it possible to write, in a relatively simple way, the general solution for the systems of differential equations in question.

After carrying out the transformations, Eqs.(3.6) reduce to the form

$$\Phi_n'''' - (p_n + q_n - \gamma_n\eta_n)\Phi_n'' + (p_nq_n - \gamma_n\eta_n\xi^2)\Phi_n = \sum_{k=0}^{n-1}[\gamma_nF_3(\ldots) + F_1''(\ldots) - q_nF_1(\ldots)], \quad \Psi_n'' - s_n\Psi_n = \sum_{k=0}^{n-1}F_2(\ldots); \quad n = 0, 1, 2, \ldots \tag{4.1}$$

After determining $\Phi_n(\xi, z)$ from the first equation of (4.1), we obtain the functions $T_n(\xi, z)$ $(n = 0, 1, 2, \ldots)$ with help of the first equation of (3.6).

Since (4.1) can be solved consecutively, it is clear that the general solution of the n-th equation can be written as follows:

$$\Phi_n(\xi, z) = \sum_{i=1}^{4} c_{nn}^{(i)}(\xi)U_{ni}(\xi, z) + \Phi_n^*(\xi, z)$$
$$\Psi_n(\xi, z) = \sum_{i=1}^{2} c_{nn}^{(i)*}(\xi)U_{ni}^*(\xi, z) + \Psi_n^*(\xi, z)$$

where $U_{ni}(\xi, z)$ and $U_{nj}^*(\xi, z)$ $(i = 1, 2, 3, 4;\ j = 1, 2)$ are the fundamental systems of solutions of the homogeneous differential equations corresponding to (4.1), $c_{nn}^{(i)}, c_{nn}^{(i)*}$ are unknown quantities, and $\Phi_n^*(\xi, z)$ and $\Psi_n^*(\xi, z)$ are particular solutions of (4.1). As we said before, the differential operator of the left-hand sides of (3.6), and hence also of (4.1), depend on n. It is therefore natural to write the particular solutions in the form of a linear combination of solutions of the previous equations:

$$\Phi_n^*(\xi, z) = \sum_{k=0}^{n-1}\sum_{i=1}^{4} c_{nk}^{(i)}U_{ki}(\xi, z), \qquad \Psi_n^*(\xi, z) = \sum_{k=0}^{n-1}\sum_{i=1}^{2} c_{nk}^{(i)*}U_{ki}^*(\xi, z)$$

The constants $c_{nk}^{(i)}$ and $c_{nk}^{(i)*}$ are found from (4.1). Taking the latter relations into account, we can write the general solution in the form

$$\Phi_n(\xi, z) = \sum_{k=0}^{n}\sum_{i=1}^{4} c_{nk}^{(i)}U_{ki}(\xi, z), \qquad \Psi_n(\xi, z) = \sum_{k=0}^{n}\sum_{i=1}^{2} c_{nk}^{(i)*}U_{ki}^*(\xi, z) \tag{4.2}$$

where $U_{ki}(\xi, z)$, $U_{ki}^*(\xi, z)$ are the fundamental systems of solutions of (4.1) as defined in /6/.

The constants $c_{nk}^{(i)}, c_{nk}^{(i)*}$ $(k = 0, 1, \ldots, n-1)$ are determined, as we have already said by substitution into (4.1). An important property of the OEP transformation is the fact, that, irrespective of the form of the right-hand sides of (4.1), i.e., irrespective of the model of the medium used, the formula for determining the constants remains the same:

$$(c_{nk}^{(i)}, c_{nk}^{(j)*}) = (-1)^{n+k}\binom{n+k+\varkappa-1}{n-k}(c_{kk}^{(i)}, c_{kk}^{(j)*}); \quad i = 1, 2, 3, 4; \quad j = 1, 2 \tag{4.3}$$

The arbitrariness remaining in the definition of $c_{nn}^{(i)}, c_{nn}^{(j)*}$ makes it possible to satisfy the boundary conditions for the quantities required.

It remains to note that everything that has been said about the transformation (2.1) for $\varkappa > 0$, applies equally to the case $\varkappa = 0$.

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ON THE STABILITY OF RODS FOR STOCHASTIC EXCITATIONS*

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The stability of motion of an elastic rod in a viscous medium compressed by a randomly acting force is studied. The conditions of stability of the rod acted upon by a stationary process with bilinear spectral density are obtained. The dependence of the statistical moments of the amplitude of the finite flexure of the rod under stationary-motion conditions on the parameters of the compressing force and the amplitude of the initial deformation is analysed. A number of problems concerning the stability of longitudinal flexure of viscoelastic constructions acted upon by random loads were discussed in /1-3/.

1. A stationary process with rational-fraction spectral density. Let us consider an elastic rod of length l, hinged at each end and compressed by forces F. The rod is in a continuous viscous medium and its equation of equilibrium has the form

$$\partial w/\partial t = -A\{EIw^{\mathrm{IV}} + [F_0 + F_1(t)]\,w''\} \tag{1.1}$$

Here A is the viscosity constant of the material of the medium, F_0, $F_1(t)$ are the deterministic (constant with respect to time) component of the compressive load, and a random oscillation with zero expectation value. The remaining notation is the generally accepted one.

Let the deflection of the rod at the initial instant be described by the sinusoid

$$w(0, x) = f_k^\circ \sin(k\pi x/l)$$

We shall seek a solution of (1.1) in the form of such a sinusoid, whose amplitude $f_k(t)$ is a solution of the equation

$$df_k/d\tau + k^4[(1-\alpha_k) - \beta_k\psi(\tau)]\,f_k = 0 \tag{1.2}$$

$$\tau = \gamma t,\quad \gamma = \frac{\pi^4 EIA}{l^4},\quad \alpha_k = \frac{F_0 l^2}{k^2\pi^2 EI},\quad \beta_k\psi(\tau) = \frac{F_1(\tau)\,l^2}{k^2\pi^2 EI}$$

(β_k is a deterministic constant).

Let us assume that the random process $\psi(\tau)$ is the result of the passage of normal "white noise" through a linear filter

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